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APPROXIMATION OF FIXED POINTS AND PROXIMAL POINT ALGORITHMS

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ABSTRACT. In this article, we give three iterative methods for approximation of fixed points of nonexpansive mappings in a Hilbert space. Then we discuss weak and strong convergence theorems for nonlinear operators of accretive and monotone type in a Hilbert space or a Banach space. In particular, we state weak and strong convergence theorems for resolvents of m-accretive operators and maximal monotone operators in a Banach space. Using these results, we also consider the convex minimization problem of finding a minimizer of a proper lower semicontinuous convex function in a Hilbert space or a Banach space.

1. INTRODUCTION

We consider the following problem: Let $f_0, f_1, f_2, \ldots, f_m$ be convex continuous functions of a Hilbert space H into $\mathbb{R}$. Then, the problem is to find a $z \in C$ such that

$$f_0(z) = \min\{f_0(x) : x \in C\}, \tag{1}$$

where $C = \{x \in H : f_1(x) \leq 0, f_2(x) \leq 0, \ldots, f_m(x) \leq 0\}$. Such a problem is called the convex minimization problem. Let us define a function $g : H \to (-\infty, \infty]$ as follows:

$$g(x) = \begin{cases} f_0(x), & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Then, g is a proper lower semicontinuous convex function and a minimizer $z \in H$ of g is a solution of the convex minimization problem (1). So, let $g : H \to (-\infty, \infty]$ be a proper convex lower semicontinuous function. Consider a convex minimization problem:

$$\min\{g(x) : x \in H\}. \tag{2}$$

For such a g, we can define a multivalued operator ∂g on H by

$$\partial g(x) = \{x^* \in H : g(y) \geq g(x) + \langle x^*, y - x\rangle, y \in H\}$$

for all $x \in H$. Such a ∂g is said to be the subdifferential of g. A monotone operator $A \subset H \times H$ is called maximal if its graph

$$G(A) = \{(x, y) : y \in Ax\}$$

is not properly contained in the graph of any other monotone operator. We know that if A is a maximal monotone operator, then $R(I + \lambda A) = H$ for all $\lambda > 0$. A monotone operator A is also called m-accretive if $R(I + \lambda A) = H$ for all $\lambda > 0$.

So, we can define, for each positive λ, the resolvent $J_\lambda : R(I+\lambda A) \to D(A)$ by $J_\lambda = (I+\lambda A)^{-1}$. We know that J_λ is a nonexpansive mapping. If $g : H \to (-\infty, \infty]$ is a proper lower semicontinuous convex function, then ∂g is a maximal monotone operator.

We know that one method for solving (2) is the proximal point algorithm first introduced by Martinet [16]. The proximal point algorithm is based on the notion of resolvent J_λ, i.e.,

$$J_\lambda x = \arg\min\left\{ g(z) + \frac{1}{2\lambda}\|z-x\|^2 : z \in H \right\}.$$

The proximal point algorithm is an iterative procedure, which starts at a point $x_1 \in H$, and generates recursively a sequence $\{x_n\}$ of points $x_{n+1} = J_{\lambda_n} x_n$, where $\{\lambda_n\}$ is a sequence of positive numbers; see, for instance, Rockafellar [26].

On the other hand, Halpern [6] and Mann [15] introduced the following iterative schemes to approximate a fixed point of a nonexpansive mapping T of H into itself:

$$x_{n+1} = \alpha_n x + (1-\alpha_n) T x_n, \quad n = 1, 2, \ldots$$

and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T x_n, \quad n = 1, 2, \ldots,$$

respectively, where $x_1 = x \in H$ and $\{\alpha_n\}$ is a sequence in $[0,1]$. Recently, Nakajo and Takahashi [18] also introduced an iterative scheme of finding a fixed point of a nonexpansive mapping in a Hilbert space by using an idea of the hybrid method in mathematical programming.

In this article, we first state three convergence theorems for nonexpansive mappings in a Hilbert space. They are convergence theorems of Halpern's type, Mann's type and Nakajo-Takahashi's type. Then, we prove a strong convergence theorem of Halpern's type and a weak convergence theorem of Mann's type for inverse-strongly-monotone mappings in a Hilbert space. In Section 6, we prove weak and strong convergence theorems for resolvents of accretive operators in a Banach space. In Section 7, we consider the strong convergence of a sequence defined by resolvents of maximal monotone operators in a Banach space. Using these results, we also discuss the convex minimization problem of finding a minimizer of a proper lower semicontinuous convex function in a Hilbert space or a Banach space.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon \right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. If E is uniformly convex, then δ satisfies that $\delta(\epsilon/r) > 0$ and

$$\left\| \frac{x+y}{2} \right\| \le r\left(1 - \delta\left(\frac{\epsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \le r$, $\|y\| \le r$ and $\|x-y\| \ge \epsilon$. Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \le \|x - y\|$ for all $y \in C$. Putting $z = P_C(x)$, we call P_C the metric projection of E onto C. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{3}$$

exists. In the case, E is called smooth. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (3) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (3) is attained uniformly for $y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single valued and uniformly norm to weak* continuous on each bounded subset of E. A Banach space E is said to satisfy Opial's condition [20] if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup y$ implies

$$\liminf_{n \to \infty} \|x_n - y\| < \liminf_{n \to \infty} \|x_n - z\|$$

for all $z \in E$ with $z \neq y$. A Hilbert space satisfies Opial's condition.

Let C be a closed convex subset of E. A mapping $T\colon C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We denote the set of all fixed points of T by $F(T)$. A closed convex subset C of E is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C into itself has a fixed point in D. Let D be a subset of E. We denote the closure of the convex hull of D by $\overline{\text{co}}D$.

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \ge 0$. If A is accretive, then we have

$$\|x_1 - x_2\| \le \|x_1 - x_2 + r(y_1 - y_2)\|$$

for all $r > 0$. An accretive operator A is said to satisfy the range condition if $\overline{D(A)} \subset \bigcap_{r>0} R(I + rA)$. If A is accretive, then we can define, for each $r > 0$, a nonexpansive single valued mapping $J_r\colon R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$. It is called the resolvent of A. We also define the Yosida approximation A_r by $A_r = (I - J_r)/r$. We know that $A_r x \in AJ_r x$ for all $x \in R(I + rA)$ and $\|A_r x\| \le \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$. We also know that for an accretive operator A satisfying the range condition, $A^{-1}0 = F(J_r)$ for all $r > 0$. An accretive operator A is said to be m-accretive if $R(I + rA) = E$ for all $r > 0$. A multi-valued operator $A\colon E \to 2^{E^*}$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \ge 0$ for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$. A monotone operator A is said to be maximal if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. The following theorems are well known; see, for instance [32].

Theorem 1. *Let E be a reflexive, strictly convex and smooth Banach space and let $A\colon E \to 2^{E^*}$ be a monotone operator. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Theorem 2. *Let E be a strictly convex and smooth Banach space and let $x, y \in E$. If $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

By Theorem 1, a monotone operator A in a Hilbert space H is maximal if and only if A is m-accretive.

3. Approximating fixed points of nonexpansive mappings

There are three iterative methods for approximation of fixed points of nonexpansive mappings in a Hilbert space which are related to the problem of finding a minimizer of a convex function.

Halpern [6] introduced the following iterative scheme to approximate a fixed point of a nonexpansive mapping in a Hilbert space. For the proof, see Wittmann [36] and Takahashi [32].

Theorem 3 ([36]). *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself that $F(T)$ is nonempty. Let P be the metric prjection of H onto $F(T)$. Let $x \in C$ and let $\{x_n\}$ be a sequence defined by $x_1 = x$ and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \dots,$$

where $\{\alpha_n\} \subset [0, 1]$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then, $\{x_n\}$ converges strongly to $Px \in F(T)$.

Mann [15] also introduced the iterative scheme for finding a fixed point of a nonexpansive mapping. For the proof, see Takahashi [32].

Theorem 4 ([15]). *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let P be the metric projection of H onto $F(T)$. Let $x \in C$ and let $\{x_n\}$ be a sequence defined by $x_1 = x$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \dots,$$

where $\{x_n\} \subset [0, 1]$ satisfies

$$0 \le \alpha_n < 1 \text{ and } \sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty.$$

Then, $\{x_n\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} Px_n$.

Recently, Nakajo and Takahashi [18] proved the following theorem for nonexpansive mappings in a Hilbert space by using an idea of the hybrid method in mathematical programming.

Theorem 5 ([18]). *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let P be the metric projection of H onto $F(T)$. Let $x_1 = x \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_1), \quad n = 1, 2, \dots, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ *satisfies* $\liminf_{n\to\infty} \alpha_n < 1$ *and* $P_{C_n \cap Q_n}$ *is the metric projection of* H *onto* $C_n \cap Q_n$. *Then,* $\{x_n\}$ *converges strongly to* $Px_1 \in F(T)$.

Shioji and Takahashi [27] extended Theorem 3 to that of a Banach space whose norm is uniformly Gâteaux differentiable. Let C and D be closed convex subsets of a Banach space E and let D be a subset of C. Then, a mapping P of C onto D is called sunny if

$$P(Px + t(x - Px)) = Px$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$.

Theorem 6 ([27]). *Let* E *be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let* C *be a nonempty closed convex subset of* E *and let* T *be a nonexpansive mapping of* C *into itself such that* $F(T)$ *is nonempty. Let* $\{\alpha_n\}$ *be a sequence of real numbers such that*

$$0 \leq \alpha_n \leq 1, \quad \lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Suppose $x_1 = x \in C$ *and* $\{x_n\}$ *is given by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \ldots.$$

Then, $\{x_n\}$ *converges strongly to* $Px \in F(T)$, *where* P *is a unique sunny nonexpansive retraction of* C *onto* $F(T)$.

Reich [22] extended also Mann's result to that of a Banach space whose norm is Fréchet differentiable.

Theorem 7 ([22]). *Let* C *be a closed convex subset of a uniformly convex Banach space* E *with a Fréchet differentiable norm, let* $T : C \to C$ *be a nonexpansive mapping such that* $F(T)$ *is nonempty, and let* $\{\alpha_n\}$ *be a real sequence such that* $0 \leq \alpha_n \leq 1$ *and* $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. *If* $x_1 = x \in C$ *and*

$$x_{n+1} = \alpha_n Tx_n + (1 - \alpha_n)x_n, \quad n = 1, 2, \ldots,$$

then $\{x_n\}$ *converges weakly to a fixed point of* T.

Problem. Is a Hilbert space in Theorem 5 replaced by a uniformly convex and smooth Banach space?

4. Approximating solutions of valiational inequalities

Let C be a closed convex subset of a Hilbert space H. Then, a mapping A of C into H is called inverse-strongly-monotone if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$; see [4] and [14]. For such a case, A is called α-inverse-strongly-monotone. If a mapping T of C into itself is nonexpansive, then $A = I - T$ is $\frac{1}{2}$-inverse-strongly-monotone and $F(T) = \mathrm{VI}(C, A)$; for example, see [8]. A mapping A of C into H is called strongly monotone if there exists a positive number η such that

$$\langle x - y, Ax - Ay \rangle \geq \eta \|x - y\|^2$$

for all $x, y \in C$. In such a case, we say that A is η-strongly monotone. If A is η-strongly monotone and k-Lipschitz continuous, i.e., $\|Ax - Ay\| \leq k\|x - y\|$ for all $x, y \in C$, then A is $\frac{\eta}{k^2}$-inverse-strongly-monotone; see [14]. Let f be a continuously Fréchet differentiable convex function H and let ∇f be the gradient of f. If ∇f is

$\frac{1}{\alpha}$-Lipschitz continuous, then ∇f is an α-inverse-strongly-monotone mapping of C into H; see [1]. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned}\|(I-\lambda A)x-(I-\lambda A)y\|^2 &= \|(x-y)-\lambda(Ax-Ay)\|^2 \\ &= \|x-y\|^2 - 2\langle x-y, Ax-Ay\rangle + \lambda^2\|Ax-Ay\|^2 \\ &\le \|x-y\|^2 + \lambda(\lambda-2\alpha)\|Ax-Ay\|^2.\end{aligned}$$

So, if $\lambda \le 2\alpha$, then $I-\lambda A$ is a nonexpansive mapping of C into H.

Theorem 8 ([7]). *Let C be a closed convex subset of a Hilbert space H. Let A be an α-invese-strongly-monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C,A) \neq \phi$. Let $x_1 = x \in C$ and let $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad n = 1, 2, \dots,$$

where $\{\alpha_n\} \subset [0,1)$ and $\{\lambda_n\} \subset [a,b] \subset (0, 2\alpha)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \ \text{and} \ \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

Then, $\{x_n\}$ converges strongly to $z = P_{F(S)\cap VI(C,A)}x$.

Theorem 9 ([34]). *Let C be a closed convex subset of a Hilbert space H. Let A be an α-inverse-strongly-monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C,A) \neq \phi$. Let $x_1 = x \in C$ and let $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad n = 1, 2, \dots,$$

where $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy

$$0 < c \le \alpha_n \le d < 1 \ \text{and} \ 0 < a \le \lambda_n \le b < 2\alpha.$$

Then, $\{x_n\}$ converges weakly to $z \in F(S) \cap VI(C,A)$.

5. Proximal point algorithms in Hilbert spaces

We consider two proximal point algorithms for sloving (2) in Section 1, with parameters $\{r_n\}$, starting at an initial point x_1 in a Hilbert space H.

Theorem 10 ([9]). *Let H be a Hilbert space and let $A \subset H \times H$ be a maximal monotone operator. Let $x_1 = x \in H$ and let $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)J_{r_n}x_n, \ n = 1, 2, \dots,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \ \text{and} \ \lim_{n\to\infty} r_n = \infty.$$

If $A^{-1}0 \neq \phi$, then $\{x_n\}$ converges strongly to $Px \in A^{-1}0$, where P is the metric projection of H onto $A^{-1}0$.

Theorem 11 ([9]). *Let H be a Hilbert space and let $A \subset H \times H$ be a maximal monotone operator. Let $x_1 = x \in H$ and let $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)J_{r_n}x_n, \ n = 1, 2, \dots,$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy* $\alpha_n \in [0,k]$ *for some* k *with* $0 < k < 1$ *and* $\lim_{n\to\infty} r_n = \infty$. *If* $A^{-1}0 \neq \phi$, *then* $\{x_n\}$ *converges weakly to* $v \in A^{-1}0$, *where* $v = \lim_{n\to\infty} Px_n$ *and* P *is the metric projection of* H *onto* $A^{-1}0$.

Using Theorems 10 and 11, we obtain the following theorems.

Theorem 12 ([9]). *Let* H *be a Hilbert space and let* $f : H \to (-\infty,\infty]$ *be a lower semicontinuous proper convex function. Let* $x_1 = x \in H$ *and let* $\{x_n\}$ *be a sequence defined by*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)J_{r_n}x_n,\ n = 1,2,\ldots,$$
$$J_{r_n}x_n = \arg\min\left\{f(z) + \frac{1}{2r_n}\|z - x_n\|^2 : z \in H\right\},$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy*

$$\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \lim_{n\to\infty} r_n = \infty.$$

If $(\partial f)^{-1}0 \neq \phi$, *then* $\{x_n\}$ *converges strongly to* $v \in H$, *which is the minimizer of* f *nearest to* x. *Further*

$$f(x_{n+1}) - f(v) \le \alpha_n(f(x) - f(v)) + \frac{1-\alpha_n}{r_n}\|J_{r_n}x_n - v\|\|J_{r_n}x_n - x_n\|.$$

Theorem 13 ([9]). *Let* H *be a Hilbert space and let* $f : H \to (-\infty,\infty]$ *be a lower semicontinuous proper convex function. Let* $x_1 = x \in H$ *and let* $\{x_n\}$ *be a sequence defined by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)J_{r_n}x_n,\ n = 1,2,\ldots,$$
$$J_{r_n}x_n = \arg\min\left\{f(z) + \frac{1}{2r_n}\|z - x_n\|^2 : z \in H\right\},$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy* $\alpha_n \in [0,k]$ *for some* k *with* $0 < k < 1$ *and* $\lim_{n\to\infty} r_n = \infty$. *If* $(\partial f)^{-1}0 \neq \phi$, *then* $\{x_n\}$ *converges weakly to* $v \in H$, *which is a minimizer of* f. *Further*

$$f(x_{n+1}) - f(v) \le \alpha_n(f(x_n) - f(v)) + \frac{1-\alpha_n}{r_n}\|J_{r_n}x_n - v\|\|J_{r_n}x_n - x_n\|.$$

Solodov and Svaiter [29] also proved the following strong convergence theorem.

Theorem 14 ([29]). *Let* H *be a Hilbert space and let* $A \subset H \times H$ *be a maximal monotone operator. Let* $x \in H$ *and let* $\{x_n\}$ *be a sequence defined by*

$$\begin{cases} x_1 = x \in H, \\ 0 = v_n + \dfrac{1}{r_n}(y_n - x_n),\ v_n \in Ay_n, \\ H_n = \{z \in H : \langle z - y_n, v_n\rangle \le 0\}, \\ W_n = \{z \in H : \langle z - x_n, x_1 - x_n\rangle \le 0\}, \\ x_{n+1} = P_{H_n\cap W_n}x_1,\ n = 1,2,\ldots, \end{cases}$$

where $\{r_n\}$ *is a sequence of positive numbers. If* $A^{-1}0 \neq \phi$ *and* $\liminf_{n\to\infty} r_n > 0$, *then* $\{x_n\}$ *converges strongly to* $P_{A^{-1}0}x_1$.

6. Convergence theorems for accretive operators

In this section, we study a strong convergence theorem of Halpern's type for accretive operators in a Banach space. We need the following lemma for the proof of our theorem.

Lemma 15 ([35]). *Let E be a reflexive Banach space whose norm is uniformly Gâteaux differentiable and let $A \subset E \times E$ be an accretive operator which satisfies the range condition. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I+rA)$. If $A^{-1}0 \neq \emptyset$, then the strong $\lim_{t\to\infty} J_t x$ exists and belongs to $A^{-1}0$ for all $x \in C$.*

See also Reich [23]. Using this result, we prove the following theorem. The proof is mainly due to Wittmann [36] and Shioji and Takahashi [27].

Theorem 16 ([10]). *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, let $A \subset E \times E$ be an accretive operator which satisfies the range condition, and let C be a nonempty closed convex subset of E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I+rA)$. Let $x_1 = x \in C$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x + (1-\alpha_n) J_{r_n} x_n, \; n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \; \sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } \lim_{n\to\infty} r_n = \infty.$$

If $A^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges strongly to an element of $A^{-1}0$.

As a direct consequence of Theorem 16, we have the following:

Theorem 17. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let $A \subset E \times E$ be an m-accretive operator. Let $x_1 = x \in E$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x + (1-\alpha_n) J_{r_n} x_n, \; n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \; \sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } \lim_{n\to\infty} r_n = \infty.$$

If $A^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges strongly to an element of $A^{-1}0$.

Next, we prove a weak convergence theorem for Mann's type for accretive operators in a Banach space. Before proving the theorem, we need the following two lemmas.

Lemma 18 ([3]). *Let C be a closed bounded convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself. If $\{x_n\}$ converges weakly to $z \in C$ and $\{x_n - Tx_n\}$ converges strongly to 0, then $Tz = z$.*

Lemma 19 ([22]). *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable, let C be a nonempty closed convex subset of E and let $\{T_0, T_1, T_2, \ldots\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=0}^{\infty} F(T_n)$ is nonempty. Let $x \in C$ and $S_n = T_n T_{n-1} \cdots T_0$ for all $n = 1, 2, \ldots$. Then the set $\bigcap_{n=0}^{\infty} \overline{\mathrm{co}}\{S_m x : m \geq n\} \cap U$ consists of at most one point, where $U = \bigcap_{n=0}^{\infty} F(T_n)$.*

For the proof of Lemma 19, see Takahashi and Kim [33]. Now we can prove the following weak convergence theorem.

Theorem 20 ([10]). *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial's condition, let $A \subset E \times E$ be an accretive operator which satisfies the range condition, and let C be a nonempty closed convex subset of E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Let $x_1 = x \in C$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \ n = 1, 2, \dots,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$\limsup_{n \to \infty} \alpha_n < 1 \text{ and } \liminf_{n \to \infty} r_n > 0.$$

If $A^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

As a direct consequence of Theorem 20, we have the following:

Theorem 21. *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial's condition and let $A \subset E \times E$ be an m-accretive operator. Let $x_1 = x \in E$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \ n = 1, 2, \dots,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$\limsup_{n \to \infty} \alpha_n < 1 \text{ and } \liminf_{n \to \infty} r_n > 0.$$

If $A^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

7. Convergence theorems for maximal monotone operators

In this section, we study strong convergence theorems for resolvents of maximal monotone operators in a Banach space. Let E be a uniformly convex and smooth Banach space and let A be a maximal monotone operator from E into E^* such that $A^{-1}0 \neq \phi$. For $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rAx_r.$$

By Theorems 1 and 2, this equation has a unique solution x_r. We denote J_r by $x_r = J_r x$ and such J_r, $r > 0$ are called resolvents of A. Now, we extend Solodov and Svaiter's result [29].

Theorem 22 ([19]). *Let E be a uniformly convex and smooth Banach space and let A be a maximal monotone operator from E into E^* such that $A^{-1}0 \neq \phi$. Suppose $\{x_n\}$ is the sequence generated by*

$$\begin{cases} x_1 \in E, \\ y_n = J_{r_n} x_n, \\ H_n = \{z \in E : \langle y_n - z, J(x_n - y_n) \rangle \geq 0\}, \\ W_n = \{z \in E : \langle x_n - z, J(x_1 - x_n) \rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_1, \ n = 1, 2, \dots, \end{cases}$$

where $\{r_n\}$ is a sequence of positive numbers. If $A^{-1}0 \neq \phi$ and $\liminf_{n \to \infty} r_n > 0$, then $\{x_n\}$ converges strongly to $P_{A^{-1}0} x_1$.

Next, we establish another extension of Solodov and Svaiter's result [29]. Before establishing it, we give a definition. Let E be a reflexive, strictly convex and smooth Banach space. The function ϕ: $E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$. Let C be a nonempty closed convex subset of E and let $x \in E$. Then there exists a unique element $x_0 \in C$ such that

$$\phi(x_0, x) = \inf\{\phi(z,x) : z \in C\}. \tag{4}$$

So, if C is a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space E and $x \in E$, we define the mapping Q_C of E onto C by $Q_C x = x_0$, where x_0 is defined by (4). It is easy to see that in a Hilbert space, the mapping Q_C is coincident with the metric projection.

Theorem 23 ([11]). *Let E be a uniformly convex and uniformly smooth Banach space and let A be a maximal monotone operator from E into E^* such that $A^{-1}0 \neq \phi$. Let $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in E, \\ y_n = Q_{r_n} x_n, \\ H_n = \{z \in E : \langle z - y_n, Jx_n - Jy_n\rangle \leq 0\}, \\ W_n = \{z \in E : \langle z - x_n, Jx_1 - Jx_n\rangle \leq 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x_1, \ n = 1, 2, \ldots, \end{cases}$$

where $\{r_n\}$ is a sequence of positive numbers such that $\liminf_{n\to\infty} r_n > 0$. *Then, $\{x_n\}$ converges strongly to $Q_{A^{-1}0}x_1$.*

Recently, Kohsaka and Takahashi [12] proved a strong convergence theorem of Halpen's type for maximal monotone operators in a Banach space.

Theorem 24 ([12]). *Let E be a smooth and uniformly convex Banach space and let $A \subset E \times E^*$ be a maximal monotone operator. Let $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and let $\{x_n\}$ be a sequence defined as follows:*

$$x_1 = x \in E,$$
$$x_{n+1} = J^{-1}(\alpha_n Jx + (1-\alpha_n) JQ_{r_n} x_n), \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \lim_{n\to\infty} r_n = \infty.$$

If $A^{-1}0 \neq \phi$, then $\{x_n\}$ converges strongly to $Q_{A^{-1}0}x$.

Probelm. If E and E^* are uniformly convex Banach spaces, does Theorem 11 hold for maximal monotone operators $A \subset E \times E^*$?

We can apply Theorems 22, 23 and 24 to find a minimizer of a convex function f. Let E be a real Banach space and let $f : E \to (-\infty, \infty]$ be a proper convex lower semicontinuous function. Then the subdifferential ∂f of f is as follows:

$$\partial f(z) = \{v \in E^* : f(y) \geq f(z) + \langle y - z, v\rangle, \forall y \in E\}, \quad \forall z \in E.$$

Theorem 25 ([19]). *Let E be a uniformly convex and smooth Banach space and let $f : E \to (-\infty, \infty]$ be a proper convex lower semicontinuous function. Assume that $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$ and let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in E \\ y_n = \arg\min_{z\in E}\{f(z) + \frac{1}{2r_n}\|z - x_n\|^2\}, \\ H_n = \{z \in E : \langle y_n - z, J(x_n - y_n\rangle \geq 0\}, \\ W_n = \{z \in E : \langle x_n - z, J(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{H_n\cap W_n} x_1, \quad n = 1, 2, \ldots. \end{cases}$$

If $(\partial f)^{-1}0 \neq \phi$, then $\{x_n\}$ converges strongly to the minimizer of f nearest to x_1.

Proof. Since $f : E \to (-\infty, \infty]$ is a proper convex lower semicontinuous function, by Rockafellar [24], the subdifferential ∂f of f is a maxmal monotone operator. We also know that

$$y_n = \arg\min_{z\in E}\left\{f(z) + \frac{1}{2r_n}\|z - x_n\|^2\right\}$$

is equivalent to

$$0 \in \partial f(y_n) + \frac{1}{r_n}J(y_n - x_n).$$

So, we have

$$0 \in J(y_n - x_n) + r_n\partial f(y_n).$$

Using Theorem 22, we get the conclusion. □

Theorem 26 ([11]). *Let E be a uniformly convex and uniformly smooth Banach space and let $f : E \to (-\infty, \infty]$ be a proper convex lower semicontinuous function. Assume that $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$ and let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in E \\ y_n = \arg\min_{z\in E}\{f(z) + \frac{1}{2r_n}\|z\|^2 - \frac{1}{r_n}\langle z, Jx_n\rangle\}, \\ 0 = v_n + \frac{1}{r_n}(Jy_n - Jx_n), \ v_n \in \partial f(y_n), \\ H_n = \{z \in E : \langle z - y_n, v_n\rangle \leq 0\}, \\ W_n = \{z \in E : \langle z - x_n, Jx_1 - Jx_n\rangle \leq 0\}, \\ x_{n+1} = Q_{H_n\cap W_n} x_1, \quad n = 1, 2, \ldots. \end{cases}$$

If $(\partial f)^{-1}0 \neq \phi$, then $\{x_n\}$ converges strongly to the minimizer of f nearest to x_1.

Proof. We also know that

$$y_n = \arg\min_{z\in E}\left\{f(z) + \frac{1}{2r_n}\|z\|^2 - \frac{1}{r_n}\langle z, Jx_n\rangle\right\}$$

is equivalent to

$$0 \in \partial f(y_n) + \frac{1}{r_n}Jy_n - \frac{1}{r_n}Jx_n.$$

So, we have $v_n \in \partial f(y_n)$ such that $0 = v_n + \frac{1}{r_n}(Jy_n - Jx_n)$. Using Theorem 23, we get the conclusion. □

Using Theorem 24, we get the following theorem.

Theorem 27 ([12]). *Let E be a smooth and uniformly convex Banach space and let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function such that $(\partial f)^{-1}0$ is nonempty. Let $\{x_n\}$ be a sequence defined as follows:*

$$x_1 = x \in E,$$
$$y_n = \arg\min_{y \in E} \left\{ f(y) + \frac{1}{2r_n}\|y\|^2 - \frac{1}{r_n}\langle y, Jx_n\rangle \right\},$$
$$x_{n+1} = J^{-1}(\alpha_n Jx + (1-\alpha_n)Jy_n), \quad n = 1, 2, \dots,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \lim_{n\to\infty} r_n = \infty.$$

Then, $\{x_n\}$ converges strongly to $Q_{(\partial f)^{-1}0}x$.

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